

A monotonicity result for the q -fractional operator

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Abstract. In this article we prove that if the q -fractional operator $({}_q\nabla_{qa}^\alpha y)(t)$ of order $0 < \alpha \leq 1$, $0 < q < 1$ and starting at some $qa \in T_q = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, $a > 0$ is positive such that $y(a) \geq 0$, then $y(t)$ is $c_q(\alpha)$ -increasing, $c_q(\alpha) = \frac{1-q^\alpha}{1-q} q^{1-\alpha}$. Conversely, if $y(t)$ is increasing and $y(a) \geq 0$, then $({}_q\nabla_{qa}^\alpha y)(t) \geq 0$. As an application, we proved a q -fractional version of the Mean-Value Theorem.

Keywords: q -fractional derivative, q -fractional integral, Caputo q -fractional derivative, $c_q(\alpha)$ -increasing.

1 Introduction and Preliminaries

Fractional calculus [1, 2, 3] has recently occupied the minds of many researchers either theoretically or in different fields of applications [4, 5]. The theory of q -fractional calculus was initiated in early of fifties of last century [6, 7, 8, 9, 10]. Then, this theory has started to be developed in the last decade or so [11]-[16]. For the preliminaries about q -fractional calculus given here shortly, we refer the reader to the survey [17] and the recent book [18]. On the other hand the theory of discrete fractional calculus started to develop rapidly specially in the last decade [19]-[27]. Very recently, some monotonicity results have been reported for fractional difference type operators of order $0 < \alpha \leq 1$ [28], and of order $1 < \alpha < 2$ [29, 30]. Motivated by what mentioned above and the fact that monotonicity results are of interest in usual calculus itself we obtain some monotonicity results for the q -fractional type operators of order $0 < \alpha \leq 1$ in Section 2. An application is also given in Section 3 by giving a q -fractional mean value theorem version.

For $0 < q < 1$, let T_q be the time scale

$$T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$$

More generally, if α is a nonnegative real number then we define the time scale

$$T_q^\alpha = \{q^{n+\alpha} : n \in \mathbb{Z}\} \cup \{0\}$$

We write $T_q^0 = T_q$.

For a function $f : T_q \rightarrow \mathbb{R}$, the nabla q -derivative of f is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in T_q - \{0\} \quad (1)$$

The nabla q -integral of f is given by

$$\int_0^t f(s) \nabla_q s = (1-q)t \sum_{i=0}^{\infty} q^i f(tq^i) \quad (2)$$

and for $0 \leq a \in T_q$

$$\int_a^t f(s) \nabla_q s = \int_0^t f(s) \nabla_q s - \int_0^a f(s) \nabla_q s$$

Alternatively, let $a = q^{n_0} \in T_q$ where $n < n_0$, the nabla q -integral of f is given by

$$\int_a^t f(s) \nabla_q s = (1-q) \sum_{i=n}^{n_0-1} q^i f(q^i) \quad (3)$$

By the fundamental theorem in q -calculus we have

$$\nabla_q \int_0^t f(s) \nabla_q s = f(t) \quad (4)$$

and if f is continuous at 0, then

$$\int_0^t \nabla_q f(s) \nabla_q s = f(t) - f(0) \quad (5)$$

Also the following identity will be helpful

$$\nabla_q \int_a^t f(t, s) \nabla_q s = \int_a^t \nabla_q f(t, s) \nabla_q s + f(qt, t) \quad (6)$$

From the theory of q -calculus and the theory of time scale more generally, the following product rule is valid

$$\nabla_q(f(t)g(t)) = f(qt)\nabla_q g(t) + (\nabla_q f(t))g(t) \quad (7)$$

The q -factorial function for $n \in \mathbb{N}$ is defined by

$$(t-s)_q^n = \prod_{i=0}^{n-1} (t - q^i s) \quad (8)$$

More generally, when α is not a positive integer, the q -factorial fractional function is defined by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{1 - \frac{s}{t} q^i}{1 - \frac{s}{t} q^{i+\alpha}} \quad (9)$$

It has the following properties

- $(t-s)_q^{\beta+\gamma} = (t-s)_q^\beta (t - q^\beta s)_q^\gamma$
- $(at - as)_q^\beta = a^\beta (t-s)_q^\beta$
- The nabla q -derivative of the q -factorial function with respect to t is

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q} (t-s)_q^{\alpha-1}$$

- The nabla q -derivative of the q -factorial function with respect to s is

$$\nabla_q(t-s)_q^\alpha = -\frac{1-q^\alpha}{1-q}(t-qs)_q^{\alpha-1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

Moreover, the q -fractional integral of order $\alpha \neq 0, -1, -2, \dots$ is defined by

$${}_qI_0^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{\alpha-1} f(s) \nabla_q s. \quad (10)$$

Let $\alpha > 0$. If $\alpha \notin \mathbb{N}$, then the α -order Caputo (left) q -fractional derivative of a function f is defined by [13]

$${}_qC_a^\alpha f(t) \triangleq {}_qI_a^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s \quad (11)$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the greatest integer less than or equal to α . If $\alpha \in \mathbb{N}$, then ${}_qC_a^\alpha f(t) \triangleq \nabla_q^n f(t)$

The following identity is useful to transform Caputo q -fractional difference equation into q -fractional integrals.

Assume $\alpha > 0$ and f is defined in suitable domains. Then [13]

$${}_qI_a^\alpha {}_qC_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a) \quad (12)$$

and if $0 < \alpha \leq 1$ then

$${}_qI_a^\alpha {}_qC_a^\alpha f(t) = f(t) - f(a) \quad (13)$$

The following identity is essential to solve linear q -fractional equations

$${}_qI_a^\alpha (x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)_q^{\mu+\alpha} \quad (0 \leq a < x < b) \quad (14)$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$.

For more about q -Gamma functions and other q -calculus concepts we refer, for example, to [17] The following lemma is used in the proof of the main result. The proof follows from (9).

Lemma 1.1. • $(1-q^i)_q^{-\alpha} = \frac{1-q^i}{1-q^{i-\alpha}} (1-q^{i+1})_q^{-\alpha}$

- $(q^{n+1}-1)_q^{-\alpha} = \frac{1-q^{-1-n}}{q^\alpha - q^{-1-n}} (q^n-1)_q^{-\alpha}$
- $(q^m - q^n)_q^{-\alpha} = \frac{1-q^{n-m}}{q^\alpha - q^{n-m}} (q^{m-1} - q^n)_q^{-\alpha}$
- $(q^m - q^{n-1})_q^{-\alpha} = \frac{1-q^{-m+n-1}}{1-q^{-m+n-1-\alpha}} (q^m - q^n)_q^{-\alpha}$

Definition 1.1. Fix $\alpha \geq 0$ and define

$$c_q(\alpha) = \frac{1-q^\alpha}{1-q} q^{1-\alpha}$$

Definition 1.2. Let $y : T_q \rightarrow R$ be a function. y is called a $c_q(\alpha)$ -increasing on T_q , if

$$y(q^{n-1}) \geq c_q(\alpha)y(q^n) \text{ for all } q^n \in T_q.$$

Definition 1.3. Let $y : T_q \rightarrow R$ be a function. y is called a $c_q(\alpha)$ -decreasing on T_q , if

$$y(q^{n-1}) \leq c_q(\alpha)y(q^n) \text{ for all } q^n \in T_q.$$

Notice that if $\alpha \geq 1$, then $c_q(\alpha) \geq 1$ and if $0 \leq \alpha \leq 1$, then $0 \leq c_q(\alpha) \leq 1$. Hence, if y is increasing (decreasing) on T_q and $0 < \alpha < 1$ then y is $c_q(\alpha)$ -increasing (decreasing) on T_q . Also, if y is $c_q(\alpha)$ -increasing (decreasing) on T_q and $\alpha > 1$, then, y is increasing (decreasing) on T_q . If $\alpha = 1$ then y is $c_q(\alpha)$ -increasing (decreasing) if and only if y is increasing (decreasing).

2 Main Results

Theorem 2.1. Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(a) \geq 0$, $a = q^{n_0} > 0$. Suppose that

$${}_q\nabla_{aq}^\alpha y(t) \geq 0 \text{ for each } t = q^n, n < n_0.$$

Then, y is $c_q(\alpha)$ -increasing on $\{t \in T_q : t \geq a = q^{n_0}\}$.

Proof. Let ${}_q\nabla_{aq}^\alpha y(t) \geq 0$ for each $t \in T_q$, $\alpha \in (0, 1)$, then

$${}_q\nabla_{aq}^\alpha y(t) = \nabla_q \nabla_{aq}^{-(1-\alpha)} y(t) = \nabla_q \left[\frac{1}{\Gamma_q(1-\alpha)} \int_{aq}^t (t - qs)_q^{-\alpha} y(s) \nabla_q(s) \right] \geq 0$$

Let $s(t) = \frac{1-q}{\Gamma_q(1-\alpha)} \sum_{i=n_0}^n q^i (q^n - q^{i+1})_q^{-\alpha} y(q^i)$. Since $\nabla_q s(t) \geq 0$, $s(t)$ is an increasing function on T_q . This implies that

$$\begin{aligned} s(q^{n_0-1}) - s(q^{n_0}) &= \frac{(1-q)q^{n_0-1}}{\Gamma_q(1-\alpha)} [(q^{n_0-1} - q^{n_0})_q^{-\alpha} y(q^{n_0-1}) \\ &+ q(q^{n_0-1} - q^{n_0+1})_q^{-\alpha} y(q^{n_0}) - q(q^{n_0} - q^{n_0+1})_q^{-\alpha} y(q^{n_0})] \\ &= \frac{(1-q)q^{n_0-1}}{\Gamma_q(1-\alpha)} [q^\alpha (q^{n_0} - q^{n_0+1})_q^{-\alpha} y(q^{n_0-1}) \\ &+ q \frac{q^\alpha - q}{1-q} (q^{n_0} - q^{n_0+1})_q^{-\alpha} y(q^{n_0}) - q(q^{n_0} - q^{n_0+1})_q^{-\alpha} y(q^{n_0})] \\ &\geq 0 \end{aligned}$$

therefore, we have

$$q \left[\frac{q^\alpha - q}{1-q} - 1 \right] y(q^{n_0}) + q^\alpha y(q^{n_0-1}) \geq 0$$

which implies that

$$y(q^{n_0-1}) \geq \frac{1-q^\alpha}{1-q} q^{1-\alpha} y(q^{n_0})$$

Now, we assume that the hypothesis is true for $n = k$. i.e.

$$y(q^{n_0-k}) \geq \frac{1-q^\alpha}{1-q} q^{1-\alpha} y(q^{n_0-k+1})$$

hence, we have

$$y(q^{n_0-k}) \geq c_q(\alpha)y(q^{n_0-k+1}) \geq c_q^2(\alpha)y(q^{n_0-k+2}) \geq \dots \geq c_q^{k-1}(\alpha)y(q^{n_0-1}) \geq c_q^k(\alpha)y(q^{n_0}) \quad (15)$$

We want to prove that

$$y(q^{n_0-k-1}) \geq c_q(\alpha)y(q^{n_0-k}). \quad (16)$$

We start by calculating,

$$\begin{aligned} & s(q^{n_0-k-1}) - s(q^{n_0-k}) \\ &= \frac{(1-q)}{\Gamma_q(1-\alpha)} \left[\sum_{i=n_0-k-1}^{n_0} q^i (q^{n_0-k-1} - q^{i+1})_q^{-\alpha} y(q^i) - \sum_{i=n_0-k}^{n_0} q^i (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^i) \right] \\ &= \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\ &+ q^{n_0-k} (q^{n_0-k-1} - q^{n_0-k+1})_q^{-\alpha} y(q^{n_0-k}) - q^{n_0-k} (q^{n_0-k} - q^{n_0-k+1})_q^{-\alpha} y(q^{n_0-k}) \\ &+ q^{n_0-k+1} (q^{n_0-k-1} - q^{n_0-k+2})_q^{-\alpha} y(q^{n_0-k+1}) - q^{n_0-k+1} (q^{n_0-k} - q^{n_0-k+2})_q^{-\alpha} y(q^{n_0-k+1}) \\ &+ \dots \\ &+ q^{n_0-1} (q^{n_0-k-1} - q^{n_0})_q^{-\alpha} y(q^{n_0-1}) - q^{n_0-1} (q^{n_0-k} - q^{n_0})_q^{-\alpha} y(q^{n_0-1}) \\ &+ q^{n_0} (q^{n_0-k-1} - q^{n_0+1})_q^{-\alpha} y(q^{n_0}) - q^{n_0} (q^{n_0-k} - q^{n_0+1})_q^{-\alpha} y(q^{n_0})] \\ &= \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\ &+ q^{n_0-k} (q^{n_0-k} - q^{n_0-k+1})_q^{-\alpha} y(q^{n_0-k}) (\frac{q^\alpha - q}{1-q} - 1) \\ &+ q^{n_0-k+1} (q^{n_0-k} - q^{n_0-k+2})_q^{-\alpha} y(q^{n_0-k+1}) (\frac{q^\alpha - q^2}{1-q^2} - 1) \\ &+ \dots \\ &+ q^{n_0-1} (q^{n_0-k} - q^{n_0})_q^{-\alpha} y(q^{n_0-1}) (\frac{q^\alpha - q^k}{1-q^k} - 1) \\ &+ q^{n_0} (q^{n_0-k} - q^{n_0+1})_q^{-\alpha} y(q^{n_0}) (\frac{q^\alpha - q^{k+1}}{1-q^{k+1}} - 1)] \end{aligned}$$

Since $s(t)$ is increasing, we get

$$\begin{aligned} & q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) + q^{n_0-k} (q^{n_0-k} - q^{n_0-k+1})_q^{-\alpha} y(q^{n_0-k}) (\frac{q^\alpha - 1}{1-q}) \\ &\geq q^{n_0-k+1} (q^{n_0-k} - q^{n_0-k+2})_q^{-\alpha} y(q^{n_0-k+1}) (\frac{1-q^\alpha}{1-q^2}) \\ &+ \dots \\ &+ q^{n_0-1} (q^{n_0-k} - q^{n_0})_q^{-\alpha} y(q^{n_0-1}) (\frac{1-q^\alpha}{1-q^k}) \\ &+ q^{n_0} (q^{n_0-k} - q^{n_0+1})_q^{-\alpha} y(q^{n_0}) (\frac{1-q^\alpha}{1-q^{k+1}}) \end{aligned}$$

Using the induction assumption (15), we get

$$\begin{aligned}
& q^{n_0-k-1}(q^{n_0-k} - q^{n_0-k+1})_q^{-\alpha} \left[q^\alpha y(q^{n_0-k-1}) + q \frac{q^\alpha - 1}{1-q} y(q^{n_0-k}) \right] \\
\geq & q^{n_0-k+1}(q^{n_0-k} - q^{n_0-k+2})_q^{-\alpha} \left(\frac{1-q^\alpha}{1-q^2} \right) (c_q(\alpha))^{k-1} y(q^{n_0}) \\
& + \dots \\
& + q^{n_0-1}(q^{n_0-k} - q^{n_0})_q^{-\alpha} \left(\frac{1-q^\alpha}{1-q^k} \right) (c_q(\alpha)) y(q^{n_0}) \\
& + q^{n_0}(q^{n_0-k} - q^{n_0+1})_q^{-\alpha} \left(\frac{1-q^\alpha}{1-q^{k+1}} \right) y(q^{n_0})
\end{aligned}$$

Since $y(q^{n_0}) \geq 0$, we conclude that $q^\alpha y(q^{n_0-k-1}) + q \frac{q^\alpha - 1}{1-q} y(q^{n_0-k}) \geq 0$ which implies that $y(q^{n_0-k-1}) \geq c_q(\alpha) y(q^{n_0-k})$ \square

Using Theorem above and the following identity that relates (Riemann) q -fractional derivative ${}_q\nabla_a^\alpha$ and the Caputo q -fractional derivative ${}_qC_a^\alpha$ of order $0 < \alpha < 1$ [19]

$$({}_qC_a^\alpha f)(t) = ({}_q\nabla_a^\alpha f)(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} y(a),$$

we can state the following Caputo monotonicity result:

Corollary 2.2. *Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(a) \geq 0$, $a = q^{n_0} > 0$. Suppose that*

$${}_qC_{aq}^\alpha y(t) \geq -\frac{(t-qa)_q^{-\alpha}}{\Gamma_q(1-\alpha)} y(qa) \text{ for each } t = q^n, n < n_0.$$

Then, y is $c_q(\alpha)$ -increasing on $\{t \in T_q : t \geq a = q^{n_0}\}$.

Theorem 2.3. *Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(q^{n_0}) \geq 0$, $a = q^{n_0}$. Suppose that y is an increasing function on T_q . Then,*

$${}_q\nabla_{aq}^\alpha y(t) \geq 0 \text{ for each } t = q^n, n < n_0.$$

Proof. We want to prove that

$${}_q\nabla_{aq}^\alpha y(t) = \nabla_q \nabla_{aq}^{-(1-\alpha)} y(t) = \nabla_q \left[\frac{1}{\Gamma_q(1-\alpha)} \int_{aq}^t (t-qs)_q^{-\alpha} y(s) \nabla_q(s) \right] \geq 0$$

Let

$$s(t) = \left[\frac{1}{\Gamma_q(1-\alpha)} \int_{aq}^t (t-qs)_q^{-\alpha} y(s) \nabla_q(s) \right] = \frac{1-q}{\Gamma_q(1-\alpha)} \sum_{i=n_0}^n q^i (q^n - q^{i+1})_q^{-\alpha} y(q^i).$$

Since $\nabla_q s(t) \geq 0$. We need to show that $s(t)$ is increasing on T_q . i.e. we need to show

that $s(q^{n_0-k-1}) - s(q^{n_0-k}) \geq 0$ for any natural number k with $k \geq 1$. In fact,

$$\begin{aligned}
& s(q^{n_0-k-1}) - s(q^{n_0-k}) \\
&= \frac{(1-q)}{\Gamma_q(1-\alpha)} \left[\sum_{i=n_0-k-1}^{n_0} q^i (q^{n_0-k-1} - q^{i+1})_q^{-\alpha} y(q^i) - \sum_{i=n_0-k}^{n_0} q^i (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^i) \right] \\
&= \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\
&+ \sum_{i=n_0-k}^{n_0} q^i (q^{n_0-k-1} - q^{i+1})_q^{-\alpha} y(q^i) - \sum_{i=n_0-k}^{n_0} q^i (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^i)] \\
&= \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\
&+ \sum_{i=n_0-k}^{n_0} q^i [\frac{q^\alpha - q^{i+1-n_0+k}}{1 - q^{i+1-n_0+k}} - 1] (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^i)] \\
&= \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\
&+ \sum_{i=n_0-k}^{n_0} q^i \frac{q^\alpha - 1}{1 - q^{i+1-n_0+k}} (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^{n_0-k-1} - q^i) \\
&- \sum_{i=n_0-k}^{n_0} q^i \frac{q^\alpha - 1}{1 - q^{i+1-n_0+k}} (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^{n_0-k-1})] \\
&\geq \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\
&+ \sum_{i=n_0-k}^{n_0} q^i \frac{1 - q^\alpha}{1 - q^{i+1-n_0+k}} (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^{n_0-k-1})] \geq 0.
\end{aligned}$$

□

Theorem 2.4. Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(q^{n_0}) > 0$, $a = q^{n_0}$. Suppose that y is a strictly increasing function on T_q . Then,

$${}_q \nabla_{aq}^\alpha y(t) > 0 \text{ for each } t = q^n, n < n_0.$$

In a similar way, the above results can be obtained for the function which takes negative value at the initial point of its domain.

Theorem 2.5. Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(q^{n_0}) \leq 0$. Suppose that

$${}_q \nabla_{aq}^\alpha y(t) \leq 0 \text{ for each } t = q^n, n < n_0.$$

Then, y is $c_q(\alpha)$ -decreasing on T_q

Theorem 2.6. Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(q^{n_0}) \leq 0$. Suppose that y is a decreasing function on T_q . Then,

$${}_q \nabla_{aq}^\alpha y(t) \leq 0 \text{ for each } t = q^n, n < n_0.$$

3 An Application

In this section, we wish to prove a Mean-Value Theorem in q -fractional calculus. First, we need the following result:

Theorem 3.1. *Let f be defined on T_q and $a, b \in T_q$ with $a < b$. Then the following equality holds:*

$${}_q\nabla_a^{-\alpha} {}_q\nabla_{aq}^\alpha f(t)|_b = f(b) - \frac{(1-q)a^{1-\alpha}(b-a)_q^{\alpha-1}(1-q)_q^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)} f(a) \text{ where } \alpha \in (0, 1).$$

Proof.

$$\begin{aligned} & {}_q\nabla_a^{-\alpha} {}_q\nabla_{aq}^\alpha f(t)|_b = {}_q\nabla_a^{-\alpha} ({}_q\nabla_a {}_q\nabla_{aq}^{-(1-\alpha)} f(t))|_b \\ &= {}_q\nabla_a {}_q\nabla_a^{-\alpha} {}_q\nabla_{aq}^{-(1-\alpha)} f(t)|_b - \frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)} {}_q\nabla_{aq}^{-(1-\alpha)} f(a)|_b \\ &= {}_q\nabla_a {}_q\nabla_a^{-\alpha} \left[{}_q\nabla_a^{-(1-\alpha)} f(t) + \frac{(1-q)a(t-qa)_q^{-\alpha} f(a)}{\Gamma_q(1-\alpha)} \right]|_b - \frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)} {}_q\nabla_{aq}^{-(1-\alpha)} f(a)|_b \\ &= {}_q\nabla_a {}_q\nabla_a^{-1} f(t)|_b - \frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)} {}_q\nabla_{aq}^{-(1-\alpha)} f(a)|_b \\ &= f(b) - \frac{(1-q)a^{1-\alpha}(b-a)_q^{\alpha-1}(1-q)_q^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)} f(a). \end{aligned}$$

□

$$\text{Let } M_q(\alpha, a, b) = \frac{(1-q)a^{1-\alpha}(b-a)_q^{\alpha-1}(1-q)_q^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}.$$

Theorem 3.2. *Assume f and g are defined on T_q , $a, b \in T_q$, $a > 0$ and g is strictly increasing on $\Delta = \{t \in T_q : t \geq a, t \leq b\}$ and satisfying $g(a) > 0$. Then, there exists $r_1, r_2 \in \Delta$ such that*

$$\frac{{}_q\nabla_{aq}^\alpha f(r_1)}{{}_q\nabla_{aq}^\alpha g(r_1)} \leq \frac{f(b) - M_q(\alpha, a, b)f(a)}{g(b) - M_q(\alpha, a, b)g(a)} \leq \frac{{}_q\nabla_{aq}^\alpha f(r_2)}{{}_q\nabla_{aq}^\alpha g(r_2)}.$$

Proof. Suppose without loss of generality to the contrary that

$$\frac{f(b) - M_q(\alpha, a, b)f(a)}{g(b) - M_q(\alpha, a, b)g(a)} > \frac{{}_q\nabla_{aq}^\alpha f(t)}{{}_q\nabla_{aq}^\alpha g(t)}$$

Since g is strictly increasing, Theorem (2.4) implies that ${}_q\nabla_{aq}^\alpha g(t) > 0$. Hence

$$\frac{f(b) - M_q(\alpha, a, b)f(a)}{g(b) - M_q(\alpha, a, b)g(a)} {}_q\nabla_{aq}^\alpha g(t) > {}_q\nabla_{aq}^\alpha f(t).$$

We apply ${}_q\nabla_a^{-\alpha}$ at $t = b$ to both sides of the above inequality and make use of Theorem 3.1 to get

$$\frac{f(b) - M_q(\alpha, a, b)f(a)}{g(b) - M_q(\alpha, a, b)g(a)} [g(b) - M_q(\alpha, a, b)g(a)] > f(b) - M_q(\alpha, a, b)f(a).$$

This leads to $g(b) > g(b)$, which is a contradiction. □

Remark 3.1. • Notice that if the edge point $a = 0$ then $M_q(\alpha, a, b) = 0$ and hence no Mean-Value Theorem.

- Notice that since $M_q(\alpha, a, b) < 1$ and $g(t)$ is strictly decreasing then $g(b) - M_q(\alpha, a, b)g(a)$ in the statement of Theorem 3.2 is not equal to zero.

References

- [1] I. Podlubny, Fractional Differential Equations, Academic Press, 1999.
- [2] S. Samko, A. A. Kilbas, Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.
- [3] A. A. Kilbas, M. H. Srivastava, J. J. Trujillo, Theory and Application of Fractional Differential Equations, North Holland Mathematics Studies 204, 2006.
- [4] R. L. Magin, Fractional Calculus in Bioengineering, Begell House, 2006.
- [5] Fatma Bozkurt, T. Abdeljawad, Hajji, M. A., Stability analysis of a fractional order differential equation model of a brain tumor growth depending on the density, Applied and Computational Mathematics , Volume: 14 Issue: 1 Pages: 50-62, 2015 .
- [6] W. Hahn, Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der Hypergeometrischen q -Differenzgleichung. Das q -Analogon der Laplace-Transformation, Math. Nachr., 2, 340-379 (1949).
- [7] R. P. Agrawal, Certain fractional q -integrals and q -derivatives, Proc. Camb. Phil. Soc. (1969), 66,365, 365-370.
- [8] W.A. Al-Salam, Some fractional q -integrals and q -derivatives, Proc. Edin. Math. Soc., vol 15 (1969), 135-140.
- [9] W.A. Al-Salam, A. Verma, A fractional Leibniz q -formula, Pacific Journal of Mathematics, vol 60, (1975), 1-9.
- [10] W. A. Al-Salam, q -Analogues of Cauchy's formula, Proc. Amer. Math. Soc. 17,182-184,(1952-1953).
- [11] M. R. Predrag,D. M. Sladana, S. S. Miomir, Fractional Integrals and Derivatives in q -calculus, Applicable Analysis and Discrete Mathematics, 1, 311-323, (2007).
- [12] F. M. Atıcı, P.W. Elloe, Fractional q -calculus on a time scale , *Journal of Non-linear Mathematical Physics* 14, 3, (2007), 333-344.
- [13] T. Abdeljawad, D. Baleanu, Caputo q -Fractional Initial Value Problems and a q -Analogue Mittag-Leffler Function, Communications in Nonlinear Science and Numerical Simulations, vol. 16 (12), 4682-4688 (2011).
- [14] Thabet Abdeljawad, J. Alzabut, The q -fractional analogue for Gronwall-type inequality, Vol. 2013 (2013), Article ID 543839, 7 pages.
- [15] Thabet Abdeljawad, Betul Benli, Dumitru Baleanu, A generalized q -Mittag-Leffler function by Caputo fractional linear equations, Abstract and Applied Analysis, vol 2012, 11 pages, Article ID 546062 (2012).
- [16] Fahd Jarad, Thabet Abdeljawad, Dumitru Baleanu. Stability of q -fractional non-autonomous systems, Nonlinear Analysis: Real and World Applications, doi: 10.1016/j.nonrwa.2012.08.001, (2012).
- [17] T. Ernst, The history of q -calculus and new method (Licentiate Thesis), U.U.D.M. Report 2000: <http://math.uu.se/thomas/Lics.pdf>.

- [18] Annaby, Mahmoud H., Mansour, Zeinab S. q-Fractional Calculus and Equations, Springer, 2012.
- [19] T. Abdeljawad, D. Dumitru, Fractional differences and integration by parts, Journal of computational Analysis and Applications, vol 13, no. 3, 574-582.
- [20] T. Abdeljawad, F. Jarad, D. Baleanu, A semigroup-like property for discrete Mittag-Leffler functions, Advances in Difference Equations, 2012/1/72.
- [21] N. Bastos, R. A. C. Ferreira, D. F. M. Torres, Discrete-Time Variational Problems, Signal Processing, Volume 91 Issue 3, March, 2011.
- [22] F. M. Atıcı, P.W. Eloe, A Transform method in discrete fractional calculus , *International Journal of Difference Equations*, vol 2, no 2, (2007), 165–176.
- [23] K.S. Miller, B. Ross, Fractional difference calculus, *Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications*, Nihon University, Koriyama, Japan, (1989), 139-152.
- [24] G. A. Anastassiou, Nabla discrete fractional calculus and nabla inequalities, Math. Comput. Modelling 51 (2010), no. 5-6, 562-571.
- [25] F. M. Atici, S. Sengul, Modeling with fractional difference equations, J. Math. Anal. Appl. 369 (2010), no. 1, 1-9.
- [26] N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, Discrete Contin. Dyn. Syst. 29 (2011), no. 2, 417-437.
- [27] H. L. Gray, N. F. Zhang, On a new definition of the fractional difference, Math. Comp. 50 (1988), no. 182, 513-529.
- [28] Analysis of discrete fractional operators, Ferhan M. Atıcı, M. Uyanik, Appl. Anal. Discrete Math. 9 (2015), 139-149.
- [29] Baoguo Jia, L. Erbe, A. Peterson, Two monotonicity results for nabla and delta fractional differences, Arch. Math. 104 (2015), 589-597.
- [30] C. S. Goodrich, Aconvexity result for for fractional differences, Appl. Math. Lett. 35 (2014), 58-62.